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New Even and Odd Nonlinear Coherent States and Their Nonclassical Properties

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Based on Roy and Roy's work (Roy, B. and Roy, P. (2000). *Journal of Optics B: Quantum Semiclass. Opt.* **2**, 65), a new type of even and odd nonlinear coherent states (NCSs) are defined. They result from Schrödinger cat states for deformed field. Using the numerical method, we study nonclassical properties of the new even and odd NCSs. It is shown that quantum statistical properties of the new even and odd NCSs are quite different from those of the usual even and odd coherent states (CSs). It is found that the squeezing only consists in the new even NCS, and the amplitude-squared squeezing and antibunching effect appear for both new even and odd NCSs in some ranges of $|\beta|$.

KEY WORDS: even and odd nonlinear coherent states; squeezing; amplitude-squared squeezing; antibunching effect.

1. INTRODUCTION

The concept of coherent states (CSs) was introduced by Glauber (1963), since then they were attained an important position in the study of quantum optics. This is because the CSs not only have physical substance but also yield a very useful representation. The usual CSs advanced by Glauber are eigenstates of the annihilation operator a of the harmonic oscillator. Based on Glauber's work, the even and odd CSs were introduced in Dodonov *et al.* (1974). The even (odd) CSs are the symmetric (antisymmetric) combination of the CSs. They are two orthonormalized eigenstates of the square (a^2) of the annihilation operator a, and essentially have two kinds of nonclassical effects: the even CS has a squeezing but no antibunching effect, while the odd CS has an antibunching but no squeezing effect (Hillery, 1987; Xia and Guo, 1989).

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On the other hand, quantum group (Drinfeld, 1986; Jimbo, 1986), introduced as a mathematical description of deformed Lie algebras, has given the possibility of generalizing the notion of creation and annihilation operators of the usual oscillator and to introduce q-oscillators (Biedenharn, 1989; Macfarlane, 1989). The latter were interpreted (Man'ko *et al.*, 1993a,b) as nonlinear oscillators with a very specific type of nonlinearity, in which the frequency of vibration depends on the energy of these vibrations through the hyperbolic cosine function containing a parameter of nonlinearity. This interpretation of q-oscillators becomes obvious if one uses the classical counterpart of the original quantum q-oscillators. This observation suggests that there might exist other types of nonlinearity for which the frequency of oscillation varies with the amplitude via a generic function f, this leads to the concept of f-oscillators devised in (Man'ko *et al.* (1997). Then, the notion of f-CSs, i.e., the nonlinear coherent states (NCSs), was straightforwardly introduced (Man'ko *et al.*, 1997), and the generation of such NCSs enter in the real possibilities of trapped systems (de Matos Filho and Vogel, 1996). These NCSs exhibit nonclassical features like squeezing and self-splitting. Based on the work (Man'ko *et al.*, 1997), the concept of even and odd NCSs were constructed in Mancini (1997). A kind of orthogonal even NCSs were introduced by Das (2000). Recently, a new kind of NCSs was constructed by Roy and Roy (2000) (referred as Roy-type NCSs hereafter). In this paper, we define a new type of even and odd NCSs. They result from Schrödinger cat states for deformed field. We investigate the quantum statistical properties of the states, including quadrature squeezing, amplitude-squared squeezing, and antibunching effect. Because of these effects they have the typical nonclassical properties. It is shown that nonclassical properties of the new even and odd NCSs are very different from those of the usual even and odd CSs.

2. DEFINITION OF NEW EVEN AND ODD NCSS

For convenience of reference and completeness, in this section we begin with some related results for the f-CSs (i.e., the NCSs) (Man'ko *et al.*, 1997) and the Roy-type NCSs (Roy and Roy, 2000).

We note that the generalized annihilation (creation) operator associated with NCSs is given by

$$A = af(N), \qquad A^{+} = f(N)a^{+}, \qquad N = a^{+}a, \tag{1}$$

where a^{+} and a are standard harmonic oscillator creation and annihilation operators and $f(x)$ is a reasonably well-behaved real function, called the nonlinearity function. From the relations (1), it follows that A, A^{+}, and N satisfy the following closed algebraic relations:

$$[N, A] = -A, \qquad [N, A^{+}] = A^{+}, \tag{2}$$

$$[A, A^+] = f^2(N)(N+1) - f^2(N-1)N. \tag{3}$$

Clearly, the nature of the nonlinear algebra depends on the choice of the nonlinearity function $f(N)$. For $f(N) = 1$ we regain the Heisenberg algebra. NCSs $|\alpha, f\rangle$ are then defined as right eigenstates of the generalized annihilation operator A (de Matos Filho and Vogel, 1996; Man'ko *et al.*, 1997):

$$A|\alpha, f\rangle = \alpha|\alpha, f\rangle. \tag{4}$$

In the number state basis, $|\alpha, f\rangle$ is given by

$$|\alpha, f\rangle = C\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}\,f(n)!}|n\rangle, \qquad C = \left\{\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n![f(n)!]^2}\right\}^{-1/2}, \tag{5}$$

where α is a complex number, $f(n)! = f(n)f(n-1)\cdots f(1)f(0)$, and $f(0) = 1$.

The canonical conjugate of the generalized annihilation and creation operators A and A^+ are given by (Roy and Roy, 2000):

$$B = a\frac{1}{f(N)}, \qquad B^+ = \frac{1}{f(N)}a^+. \tag{6}$$

Thus A, B^+, and their conjugates satisfy the following algebras (Roy and Roy, 2000):

$$[A, B^+] = 1, \qquad [B, A^+] = 1. \tag{7}$$

In the number state basis, the Roy-type NCSs (Roy and Roy, 2000) are defined as the right eigenstates of the new generalized annihilation operator B,

$$|\beta, f\rangle = N_f\sum_{n=0}^{\infty}\frac{\beta^n f(n)!}{\sqrt{n!}}|n\rangle, \qquad N_f = \left\{\sum_{n=0}^{\infty}\frac{|\beta|^{2n}[f(n)!]^2}{n!}\right\}^{-1/2}, \tag{8}$$

where β is an arbitrary complex number.

Now, we follow the definition of the even and odd NCSs (Mancini, 1997) (i.e., the eigenstates of the operator A^2), and define a new kind of even (+) and odd (−) NCSs (called even/odd Roy-type NCSs) in a straightforward manner as

$$|\beta, f\rangle_\pm = N_\pm(|\beta, f\rangle \pm |-\beta, f\rangle), \tag{9}$$

where the constants $N_\pm$ are determined from the normalization condition ${}_\pm\langle\beta, f \mid \beta, f\rangle_\pm = 1$, and the result is

$$N_\pm = \left\{2 \pm 2N_f^2\sum_{n=0}^{\infty}\frac{(-|\beta|^2)^n[f(n)!]^2}{n!}\right\}^{-1/2}. \tag{10}$$

It is easy to prove that the states (9) are the orthonormalized eigenstates of the square B^2 of the operator B, i.e.,

$$B^2|\beta,\ f\rangle_{\pm} = \beta^2|\beta,\ f\rangle_{\pm}. \tag{11}$$

They can be considered as Schrödinger cat states for a deformed field.

By using Eqs. (8) and (9), the even and odd Roy-type NCSs can be expanded in the number basis as

$$|\beta,\ f\rangle_{\pm} = N_{\pm}N_f \sum_{n=0}^{\infty} \frac{[\beta^n \pm (-\beta)^n]\, f(n)!}{\sqrt{n!}} |n\rangle. \tag{12}$$

From this expression, it becomes evident that the even (odd) Roy-type NCS has a vanishing probability of containing an odd (even) number of photons. As a consequence the number probability is found to be strongly oscillating, which is a peculiarity of highly nonclassical states. These oscillations are also present in the nondeformed even and odd CSs, but here the profile of the distribution will be determined by f.

3. NONCLASSICAL PROPERTIES OF THE EVEN AND ODD ROY-TYPE NCSS

In this section, we shall examine squeezing, amplitude-squared squeezing, and antibunching properties of the even and odd Roy-type NCSs $|\beta,\ f\rangle_{\pm}$ given by Eq. (12). However, before we proceed any further it is necessary to specify the nonlinearity function $f(n)$. From Eq. (12), it is clear that for every choice of $f(n)$ we shall have the different even and odd NCSs. In the present case, we choose the following nonlinearity function which has been used in the description of the motion of a trapped ion (de Matos Filho and Vogel, 1996):

$$f(n) = L_n^1(\eta^2)[(n+1)L_n^0(\eta^2)]^{-1}, \tag{13}$$

where η is known as the Lamb-Dicke parameter and $L_n^m(x)$ are generalized Laguerre polynomials (Abramowitz and Stegun, 1972). Clearly, $f(n) = 1$ when $\eta = 0$ and in this case the new even and odd NCSs become the usual even and odd CSs (Hillery, 1987; Xia and Guo, 1989) respectively. However, when $\eta \neq 0$ nonlinearity starts developing, with the degree of nonlinearity depending on the magnitude of the parameter η.

3.1. Quadrature Squeezing

We first consider the usual squeezing in terms of the quadrature operators X_1 and X_2 defined as

$$X_1 = (a^+ + a)/2, \qquad X_2 = i(a^+ - a)/2, \tag{14}$$

such that

$$[X_1, X_2] = i/2, \tag{15}$$

which implies the uncertainty relation

$$\left\langle(\Delta X_1)^2\right\rangle\left\langle(\Delta X_2)^2\right\rangle \geq \frac{1}{16}. \tag{16}$$

Then the squeezing of the two operators may be conveniently described by the following two parameters

$$D_i(1) = 4\left\langle(\Delta X_i)^2\right\rangle - 1, \quad (i = 1, 2). \tag{17}$$

The quantities given in Eq. (17) are also described in terms of the operators a and a^+ as follows:

$$D_1(1) = 2\langle a^+ a\rangle + \left\langle a^{+^2} + a^2\right\rangle - \langle a^+ + a\rangle^2, \tag{18a}$$

$$D_2(1) = 2\langle a^+ a\rangle - \left\langle a^{+^2} + a^2\right\rangle + \langle a^+ - a\rangle^2, \tag{18b}$$

where $-1 \leq D_i(1) < 0$ for the usual squeezing of the field in the direction $X_i(i = 1$ or 2), and the maximum squeezing (100%) is obtained when $D_i(1) = -1(i = 1$ ior 2).

Now, we study the characteristics of the squeezing in the even and odd Roy-type NCSs [i.e., the states described by Eq. (12)].

Using Eq. (12), for the even and odd Roy-type NCSs, we have obtained the following expectation values of some operators:

$$\langle a^+ a\rangle_\pm = N_\pm^2 N_f^2 \sum_{n=0}^{\infty} \frac{|\beta^{n+1} \pm (-\beta)^{n+1}|^2 [f(n+1)!]^2}{n!}, \tag{19}$$

$$\langle a\rangle_\pm = 0, \quad \langle a^+\rangle_\pm = 0, \tag{20}$$

$$\beta^{-2}\langle a^2\rangle_\pm = \beta^{*^{-2}}\langle a^{+^2}\rangle_\pm = N_\pm^2 N_f^2 \sum_{n=0}^{\infty} \frac{|\beta^n \pm (-\beta)^n|^2 f(n)! f(n+2)!}{n!}. \tag{21}$$

Substituting Eqs. (19)–(21) into Eq. (18), with the aid of a numerical method, the variations of functions $D_1(1)$ and $D_2(1)$ versus $|\beta|$ (when arg $\beta = 0$ and $\eta^2 = 0.7$) are shown in detail in Fig. 1.

From Fig. 1, it is evident that in some ranges of $|\beta|$, the quadrature squeezing only consists in the even Roy-type NCS (in the direction X_1). For a fixed value ($\eta^2 = 0.7$) of the parameter η, the ranges are $0.4323 < |\beta|$. The result shows that the quadrature squeezing properties of the even Roy-type NCS are very different from those of the usual even CS (Xia and Guo, 1989).

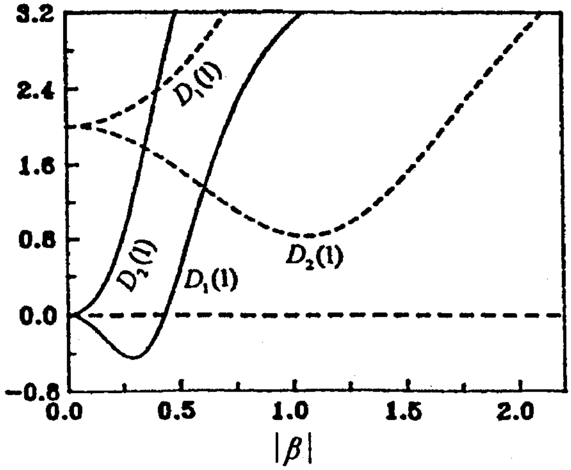

Fig. 1. Variation of the functions $D_1(1)$ and $D_2(1)$ with $|\beta|$ for $\eta^2 = 0.7$. Solid curves (the even Roy-type NCS) and broken curves (the odd Roy-type NCS).

3.2. Amplitude-Squared Squeezing

To examine whether or not the even and odd Roy-type NCSs exhibit amplitude-squared squeezing, we introduce the following Hermitian operators:

$$Y_1 = \left(a^{+^2} + a^2\right)/2, \qquad Y_2 = i\left(a^{+^2} - a^2\right)/2. \tag{22}$$

Then Y_1 and Y_2 obey the commutation relation

$$[Y_1, Y_2] = \frac{i}{2}\left[a^2, a^{+^2}\right] = i(2N+1), \quad (N = a^+a), \tag{23}$$

and the uncertainty relation

$$\left\langle(\Delta Y_1)^2\right\rangle\left\langle(\Delta Y_2)^2\right\rangle \geq |\langle N + 1/2\rangle|^2. \tag{24}$$

A state is said to show amplitude-squared squeezing in the Y_i variable if

$$\left\langle(\Delta Y_i)^2\right\rangle < |\langle N + 1/2\rangle|, \quad (i = 1, 2). \tag{25}$$

To examine the degree of the amplitude-squared squeezing, we define the squeezed degree of the amplitude-squared squeezing in the following form

$$D_i(2) = \frac{\left\langle(\Delta Y_i)^2\right\rangle - \frac{1}{4}\left\langle\left[a^2, a^{+^2}\right]\right\rangle}{\frac{1}{4}\left\langle\left[a^2, a^{+^2}\right]\right\rangle}, \quad (i = 1, 2), \tag{26}$$

which is also described in terms of a and a^+ as follows:

$$D_1(2) = \frac{2\langle a^{+^2}a^2\rangle + \langle a^{+^4} + a^4\rangle - \langle a^{+^2} + a^2\rangle^2}{\langle a^2a^{+^2}\rangle - \langle a^{+^2}a^2\rangle}, \tag{27a}$$

$$D_2(2) = \frac{2\langle a^{+^2}a^2\rangle - \langle a^{+^4} + a^4\rangle + \langle a^{+^2} - a^2\rangle^2}{\langle a^2a^{+^2}\rangle - \langle a^{+^2}a^2\rangle}, \tag{27b}$$

where $-1 \leq D_i(2) < 0$ for the amplitude-squared squeezing of the field in the direction $Y_i (i = 1 \text{ or } 2)$, and the maximum squeezing (100%) is obtained when $D_i(2) = -1 (i = 1 \text{ or } 2)$.

Using Eq. (12), for the even and odd Roy-type NCSs, we have obtained the following expectation values of some operators:

$$\langle a^{+^2}a^2\rangle_\pm = N_\pm^2 N_f^2 \sum_{n=0}^{\infty} \frac{|\beta^{n+2} \pm (-\beta)^{n+2}|^2[f(n+2)!]^2}{n!}. \tag{28}$$

$$\beta^{-4}\langle a^4\rangle_\pm = \beta^{*^{-4}}\langle a^{+^4}\rangle_\pm = N_\pm^2 N_f^2 \sum_{n=0}^{\infty} \frac{|\beta^n \pm (-\beta)^n|^2 f(n)! f(n+4)!}{n!}. \tag{29}$$

$$\langle a^2a^{+^2}\rangle_\pm = N_\pm^2 N_f^2 \sum_{n=0}^{\infty} \frac{|\beta^n \pm (-\beta)^n|^2[f(n)!]^2}{n!}(n+1)(n+2). \tag{30}$$

Substituting Eqs. (21), (28)–(30) into Eq. (27), with the aid of a numerical method, the variations of functions $D_1(2)$ and $D_2(2)$ versus $|\beta|$ (when arg $\beta = 0$ and $\eta^2 = 0.7$) are shown in detail in Fig. 2.

From Fig. 2, we can obtain that for a fixed value of the parameter η and in some different ranges of $|\beta|$, the even and odd Roy-type NCSs may exhibit the amplitude-squared squeezing in the direction X_1. For example, the corresponding ranges for $\eta^2 = 0.7$ are $|\beta| < 0.5135$ and $|\beta| < 1.5196$, respectively. This means that the amplitude-squared squeezing properties of the even and odd Roy-type NCSs are very different from those of the usual even and odd CSs (Sun *et al.*, 1992; Xia and Guo, 1989).

3.3. Antibunching Effect

Now, we study the antibunching effect of the even and odd Roy-type NCSs given by Eq. (12). If the second-order correlation function of a light field (Walls, 1983) is less than 1, i.e., $g^{(2)}(0) < 1$, one says that the light field exhibits an antibunching effect. In a similar way, we introduce the second-order correlation for the even and odd Roy-type NCSs,

$$g_\pm^{(2)}(0) = \frac{{}_\pm\langle \beta, f \mid a^{+^2}a^2 \mid \beta, f\rangle_\pm}{{}_\pm\langle \beta, f \mid a^+a \mid \beta, f\rangle_\pm^2}. \tag{31}$$

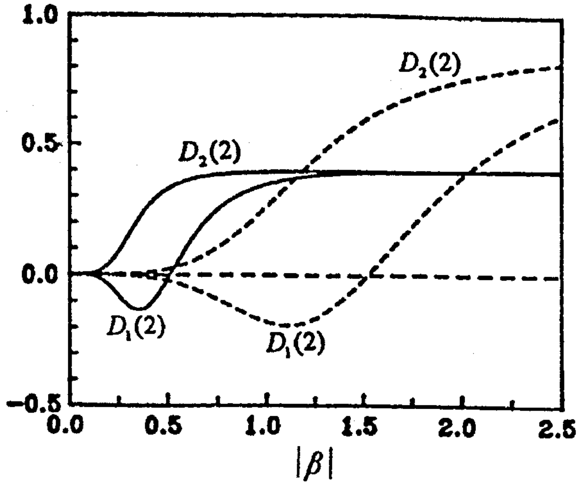

Fig. 2. Variation of the functions $D_1(2)$ and $D_2(2)$ with $|\beta|$ for $\eta^2 = 0.7$. Solid curves (the even Roy-type NCS) and broken curves (the odd Roy-type NCS).

If $g_+^{(2)}(0) < 1$ (or $g_-^{(2)}(0) < 1$), we say that the even (or odd) Roy-type NCS given by Eq. (12) exhibits the antibunching effect.

Substituting Eqs. (19) and (28) into Eq. (31), for the even and odd Roy-type NCSs, we have

$$g_\pm^{(2)}(0) = \frac{\sum_{n=0}^{\infty} \frac{|\beta^{n+2} \pm (-\beta)^{n+2}|^2 [f(n+2)!]^2}{n!}}{N_\pm^2 N_f^2 \left(\sum_{n=0}^{\infty} \frac{|\beta^{n+1} \pm (-\beta)^{n+1}|^2 [f(n+1)!]^2}{n!} \right)^2}. \tag{32}$$

The right hand side of Eq. (14) can be less or greater than 1, producing either bunching (super-Poissonian statistics) or antibunching (sub-Poissonian statistics) independently of the symmetry of the state, in contrast with the usual algebra where antibunching effects are shown only by odd CSs (Xia and Guo, 1989). With the aid of a numerical method, the variations of functions $g_\pm^{(2)}(2)$ versus $|\beta|$ (when $\arg\beta = 0$ and $\eta^2 = 0.7$) are shown in detail in Fig. 3.

From Fig. 3, we can see that the even Roy-type NCS may exhibit the antibunching effect in some wide ranges of $|\beta|$ for a fixed value of η. For example, for $\eta^2 = 0.7$, except the ranges $4.3250 < |\beta| < 5.7799$ and $14.3362 < |\beta| < 17.0509$, the even Roy-type NCS exhibits the antibunching effect. While the odd Roy-type NCS has no antibunching effect in some ranges of $|\beta|$ for a fixed value of η. For example, the corresponding ranges for $\eta^2 = 0.7$ are $4.5133 < |\beta| < 5.9436$ and $14.6557 < |\beta| < 17.0561$. This means that the sub-Poissonian distributions

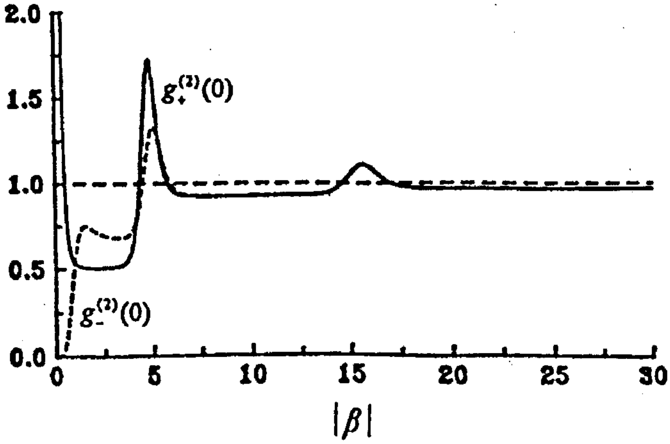

Fig. 3. Variation of the function $g_\pm^{(2)}(0)$ with $|\beta|$ for $\eta^2 = 0.7$. Solid curve (the even Roy-type NCS) and broken curve (the odd Roy-type NCS).

(i.e., the antibunching properties) of the even and odd Roy-type NCSs are very different from those of the usual even and odd CSs (Xia and Guo, 1989).

4. CONCLUSIONS

To sum up, we introduced a type of new even and odd NCSs (the even and odd Roy-type NCSs), which has rather different statistical properties from those of the usual even and odd CSs. It is found that for a fixed value ($\eta^2 = 0.7$) of the parameter η, the squeezing only consists in the even Roy-type NCS, and the amplitude-squared squeezing and antibunching effect appear for both even and odd Roy-type NCSs in some different ranges of $|\beta|$.

It is interesting to note that when $f(n) \to 1$, the even and odd Roy-type NCSs become the usual even and odd CSs. Therefore, the usual even and odd CSs are the special cases of the new even and odd NCSs when $f(n) \to 1$.

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